MATH 1104 LINEAR ALGEBRA

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 5

- Complex Numbers.
- De Moivre's Theorem.
- Roots of a Complex Number.

COMPLEX NUMBERS

A complex number z is of the form

$$z = a + ib, \text{ where } i^2 = -1, \text{ and } a, b \in R.$$

a=real part of z, a =Re z.
b=imaginary part of z, b =Im z.
z is real $\iff b = 0$.
z is purely imaginary $\iff a = 0$.

Let $z = a + ib$ and $w = c + id$. Then,

$$z + w = a + c + i(b + d)$$

$$z - w = a - c + i(b - d)$$

$$z \cdot w = (a + ib) \cdot (c + id) = ac - bd + i(ad + bc).$$

$$kz = ka + i(kb), \; k \in R.$$

z=a+ib	$iz = ia - b$
$\lvert z\rvert = \sqrt{a^2 + b^2}$	$\lvert iz\rvert = \sqrt{a^2 + b^2}$
$\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}}$	$\cos\phi = \dfrac{-b}{\sqrt{a^2 + b^2}}$
$\sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$	$\sin\phi = \dfrac{a}{\sqrt{a^2 + b^2}}$

$\cos\theta = \sin\phi$ and $\sin\theta = -\cos\phi$.
Multiplying z by i rotates z counter clockwise by 90^0.

$$z = a + ib \Longrightarrow \overline{z} = a - ib, \quad (\overline{z} \text{ is the complex conjugate of } z).$$

Properties of the complex conjugate:

$\overline{z + w} = \overline{z} + \overline{w}$ $\qquad$ $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

$\overline{z - w} = \overline{z} - \overline{w}$ $\qquad$ $z + \overline{z} = 2 \text{ Re } z$

$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ $\qquad$ $z - \overline{z} = 2i \text{ Im } z$

$\overline{\overline{z}} = z$

Let $z = a + ib$ and $w = c + id \neq 0$.

$$\begin{aligned}\frac{z}{w} &= \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{ac+bd+i(bc-ad)}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} = x + iy\end{aligned}$$

where $x = \dfrac{ac+bd}{c^2+d^2}$ and $y = \dfrac{bc-ad}{c^2+d^2}$.

The absolute value (modulus) of $z = a + ib$ is

$$|z| = \sqrt{z\,\bar{z}} = \sqrt{a^2+b^2}$$

We have the following equalities:

$$|zw| = |z| \cdot |w|, \quad z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$$

Example: Let $z = 9 - 8i$ and $w = 5 + 2i$. Then find $|z|$, $|w|$, $|z/w|$.

Write $\dfrac{z}{w}$ in the form of $a + ib$.

Solution:

$$|z| = \sqrt{9^2 + (-8)^2} = \sqrt{145}$$

$$|w| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$|z/w| = \frac{|z|}{|w|} = \frac{\sqrt{145}}{\sqrt{29}} = \frac{\sqrt{5 \cdot 29}}{\sqrt{29}} = \sqrt{5}$$

$$\frac{z}{w} = \frac{9-8i}{5+2i} = \frac{9-8i}{5+2i} \cdot \frac{5-2i}{5-2i} = \frac{(45-16)+i(-40-18)}{25+4} = 1 - 2i.$$

Homework:

1. Let $z = 3 + 4i$ and $w = 5 - 2i$. Express the followings in the form of $a + ib$.
   $(z-w)^2, \quad \frac{z}{w}, \quad \frac{\overline{z}}{\overline{w}}, \quad \frac{1}{z^2}, \quad \frac{w}{2z}.$
2. Find: $\operatorname{Re} \frac{1}{2+i}, \quad \operatorname{Im} \frac{2+i}{3+4i}, \quad \operatorname{Im} \frac{2-i}{3-4i}.$
3. Write the followings in the form of $a + ib$.
   $\frac{11+2i}{4+3i}, \quad (3+5i)(3-5i), \quad \frac{6+i}{7+3i},$
   $(7-3i)-(-2+4i), \quad \frac{1}{(3+4i)^2}, \quad \frac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)}.$
4. Solve for z if:
   $iz = 2 - i, \quad (4-3i)\overline{z} = i.$
5. If $z = 1 - 5i$ and $w = 3 + 4i$, find
   $|z|$, $|w|$, $|z/w|$, $|\overline{z/w}|$, and $|\overline{z}/\overline{w}|$,

Trigonometric Ratios of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	–

(2π radian is $360°$ or 1 radian$= \frac{180^0}{\pi} = 57^0$).

Polar Form of a Complex Number

Let $z = a + ib$.

$$\begin{aligned}\cos\theta &= \frac{a}{|z|} \Longrightarrow a = |z|\cos\theta \\ \sin\theta &= \frac{b}{|z|} \Longrightarrow b = |z|\sin\theta \\ z &= a + ib = |z|\cos\theta + i|z|\sin\theta = |z|(\cos\theta + i\sin\theta) = |z|cis\theta.\end{aligned}$$

Here θ is the angle between the positive real axis and the point z, $-\pi < \theta \leq \pi$ (all angles are measured in radians).
θ is called the argument of z, and it is denoted by $\theta =$arg$z$.

$$\boxed{z = |z|(\cos\theta + i\sin\theta)}$$

is called the polar form of z.

Example: Find the polar form of $z = 1 + i$.

Solution: $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\left.\begin{aligned}\cos\theta = \frac{a}{|z|} = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{b}{|z|} = \frac{1}{\sqrt{2}}\end{aligned}\right\} \Longrightarrow \theta = \pi/4 \Longrightarrow z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4).$$

Example: What is the polar form of $z = 3 + i3\sqrt{3}$?

Solution: $|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6.$

$$\left.\begin{aligned}\cos\theta = \frac{a}{|z|} = \frac{3}{6} = \frac{1}{2} \\ \sin\theta = \frac{b}{|z|} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}\end{aligned}\right\} \Longrightarrow \theta = \pi/3 \Longrightarrow z = 6(\cos\pi/3 + i\sin\pi/3).$$

Example: What is the polar form of $z = \sqrt{2} - i\sqrt{2}$?

Solution: $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = 2.$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{|z|} = \frac{\sqrt{2}}{2} \\ \sin\theta &= \frac{b}{|z|} = \frac{-\sqrt{2}}{2} \end{aligned}\right\} \Longrightarrow \theta = -\pi/4 \Longrightarrow z = 2(\cos\pi/4 - i\sin\pi/4).$$

Homework:

1. Write the polar form of the following complex numbers:

   $z = -4 + 4i, \qquad z = 4i, \qquad z = -7, \qquad z = 1, \qquad z = \dfrac{2+2i}{1-i}.$

2. Represent in the form of $a + ib$:

   $z = 4(\cos\pi/2 + i\sin\pi/2), \qquad z = \sqrt{8}(\cos\pi/4 + i\sin\pi/4),$

   $z = 2\mathrm{c}i\mathrm{s}(-\pi/6), \qquad z = \dfrac{2\mathrm{c}i\mathrm{s}(-3\pi/4)}{2\mathrm{c}i\mathrm{s}(5\pi/6)}.$

Complex Multiplication and Division in Polar Form

Let $z_1 = |z_1|\mathrm{c}i\mathrm{s}\theta_1$ and $z_2 = |z_2|\mathrm{c}i\mathrm{s}\theta_2$.

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot \mathrm{c}i\mathrm{s}(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2),$$

$$\overline{z_1} = r_1 \text{ cis } (-\theta_1)$$

Example: Let $z = 2\mathrm{c}i\mathrm{s}\dfrac{3\pi}{8}$ and $w = 5\mathrm{c}i\mathrm{s}\dfrac{2\pi}{3}$.
Then,

$$\overline{z} = 2\mathrm{c}i\mathrm{s}\frac{-3\pi}{8} \text{ and } \overline{w} = 5\mathrm{c}i\mathrm{s}\frac{-2\pi}{3},$$

$$z \cdot w = \left(2\mathrm{c}i\mathrm{s}\frac{3\pi}{8}\right)\left(5\mathrm{c}i\mathrm{s}\frac{2\pi}{3}\right) = 2 \cdot 5\mathrm{c}i\mathrm{s}(\frac{3\pi}{8} + \frac{2\pi}{3}) = 10\mathrm{c}i\mathrm{s}(\frac{25\pi}{24}),$$

$$\frac{z}{w} = \frac{2}{5}\mathrm{c}i\mathrm{s}\left(\frac{3\pi}{8} - \frac{2\pi}{3}\right) = \frac{2}{5}\mathrm{c}i\mathrm{s}\left(\frac{-7\pi}{24}\right).$$

Example: $z = \mathrm{c}i\mathrm{s}\left(\frac{\pi}{2}\right)$ and $w = 2\mathrm{c}i\mathrm{s}\left(\frac{-\pi}{3}\right)$. Find $z \cdot w$, z/w, $\overline{z}$ and $\overline{w}$ and write them in standard form.
Solution:

$$\begin{aligned}
z \cdot w &= \mathrm{c}i\mathrm{s}\left(\frac{\pi}{2}\right) \cdot 2\mathrm{c}i\mathrm{s}\left(\frac{-\pi}{3}\right) = 2\mathrm{c}i\mathrm{s}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) \\
&= 2\mathrm{c}i\mathrm{s}\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i. \\
\frac{z}{w} &= \frac{\mathrm{c}i\mathrm{s}\left(\frac{\pi}{2}\right)}{2\mathrm{c}i\mathrm{s}\left(\frac{-\pi}{3}\right)} = \frac{1}{2}\mathrm{c}i\mathrm{s}\left(\frac{\pi}{2} - \frac{-\pi}{3}\right) \\
&= \frac{1}{2}\mathrm{c}i\mathrm{s}\left(\frac{5\pi}{6}\right) = \frac{1}{2}\left(\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{-\sqrt{3}}{4} + i\frac{1}{4}. \\
\overline{z} &= \mathrm{c}i\mathrm{s}\left(\frac{-\pi}{2}\right) = -i. \\
\overline{w} &= 2\mathrm{c}i\mathrm{s}\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}.
\end{aligned}$$

De Moivre's Theorem:

$\boxed{z^n = (|z|(\cos\theta + i\sin\theta))^n = |z|^n(\cos n\theta + i\sin n\theta), \text{ for any positive integer } n}$.

Example: Write $z = (1+i)^{20}$ in the form of $a + ib$.
Solution: $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

$$\begin{aligned}
(1+i)^{20} &= (\sqrt{2})^{20}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right) = 2^{10}(\cos(5\pi) + i\sin(5\pi)) \\
&= 2^{10}(\cos\pi + i\sin\pi) = 2^{10}(-1 + i\cdot 0) = -2^{10} = -1024.
\end{aligned}$$

$$(1+i)^2 = 2i \Longrightarrow (1+i)^4 = (2i)^2 = -4.$$

$$(1+i)^{20} = ((1+i)^4)^5 = (-4)^5 = -2^{10}.$$

Homework: Express the following complex numbers in the form of $a + ib$.

1. $z = (2cis(\pi/3))^6$. Ans: 64.
2. $z = (-1 + i)^4$.
3. $z = (1 - i)^{10}$. Ans: $-32i$
4. $z = (1 - i)^{27}$. Ans: $-2^{13}(1 + i)$
5. $z = (1 + i)^{12}$. Ans: -64
6. $z = (1 - i)^6(\sqrt{3} + i)^3$. Ans: -64
7. $z = (\sqrt{3} - i)^9(2 - 2i)^5$. Ans: $-65536(1 + i)$.

Roots of a Complex Number

$$\boxed{z^n = \alpha\ cis\theta \Longrightarrow z_k = \sqrt[n]{\alpha}\ cis\left(\frac{\theta + 2k\pi}{n}\right), \text{ where } k = 0, 1, 2, ..., n - 1.}$$

Example: Find all fourth roots of 1.

Solution:

$$z^4 = 1 = \cos 0 + i \sin 0 \Longrightarrow z_k = \text{cis}\left(\frac{0 + 2k\pi}{4}\right), k = 0, 1, 2, 3.$$

$$z_0 = cis0 = 1, \quad z_1 = cis\frac{\pi}{2} = i, \quad z_2 = cis\pi = -1, \quad z_3 = cis\frac{3\pi}{2} = -i.$$

Example: Find all fourt roots of i.

Solution:

$$z^4 = i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} \Longrightarrow z_k = \text{cis}\left(\frac{\frac{\pi}{2} + 2k\pi}{4}\right), k = 0, 1, 2, 3.$$

$$z_0 = cis\left(\frac{\pi}{8}\right), \quad z_1 = cis\left(\frac{5\pi}{8}\right), \quad z_2 = cis\left(\frac{9\pi}{8}\right), \quad z_3 = cis\left(\frac{13\pi}{8}\right).$$

Example: Let $z^3 = -8i$. Find z and write it in the standard form.
Solution: $\alpha = |-8i| = 8$, $\theta = -\pi/2$.

$$z_k = \sqrt[3]{8}\ \text{cis}\left(\frac{-\pi/2 + 2k\pi}{3}\right);\ k = 0, 1, 2.$$

$$\begin{aligned}
z_0 &= 2\ \text{cis}\left(\frac{-\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{-1}{2}\right) = \sqrt{3} - i. \\
z_1 &= 2\ \text{cis}\,(\pi/2) = 2(0 + i) = 2i. \\
z_2 &= 2\ \text{cis}\,(7\pi/6) = 2\left(\frac{-\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i.
\end{aligned}$$

Example: Find the roots of $z^2 + z + 1 = 0$.
Solution:

$$z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = 0 \Longrightarrow \left(z + \frac{1}{2}\right)^2 = \frac{-3}{4}$$

$$\Longrightarrow z + \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i \Longrightarrow z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Homework: Find all complex numbers such that:

1. $z^2 = i$
2. $z^3 = i$
3. $z^3 = -1$
4. $z^3 = 27i$
5. $z^3 = -27i$
6. $z^3 = 64i$
7. $z^4 = -1$
8. $z^4 = 2(i\sqrt{3} - 1)$
9. $z^6 = -64$

Answers:

1. $z_0 = \dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$, $z_1 = -\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$.

2. $z_0 = \dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$, $z_1 = \dfrac{-\sqrt{3}}{2} + i\dfrac{1}{2}$, $z_2 = -i$.

3. $z_0 = i$, $z_1 = -\dfrac{\sqrt{3}}{2} - i\dfrac{1}{2}$, $z_2 = \dfrac{\sqrt{3}}{2} - i\dfrac{1}{2}$.

4. $z_0 = -3i$, $z_1 = -\dfrac{3\sqrt{3}}{2} + i\dfrac{3}{2}$, $z_2 = \dfrac{3\sqrt{3}}{2} + i\dfrac{3}{2}$.

5. $z_0 = 3i$, $z_1 = -\dfrac{3\sqrt{3}}{2} - i\dfrac{3}{2}$, $z_2 = \dfrac{3\sqrt{3}}{2} - i\dfrac{3}{2}$.

6. $z_0 = -4i$, $z_1 = -2\sqrt{3} + 2i$, $z_2 = 2\sqrt{3} + 2i$.

8. $\sqrt{2}\ e^{\pi i/6}$, $\sqrt{2}\ e^{4\pi i/6}$, $\sqrt{2}\ e^{7\pi i/6}$, $\sqrt{2}\ e^{10\pi i/6}$.

Note that $e^{i\theta} = \cos\theta + i\sin\theta$.